Phase Transitions of Disordered Travelling Salesperson Problems solved with Linear Programming

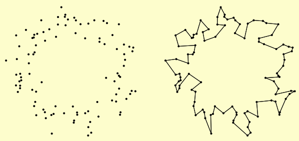
H. Schawe and A.K. Hartmann
Institut für Physik, Universität Oldenburg, Germany

The Traveling Salesperson Problem (TSP) [1]

1

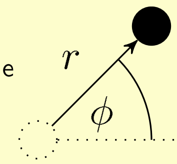
Given a set of N cities V and their pairwise distances c_{ij} with $i, j \in V$, find the shortest cyclic tour visiting all cities.

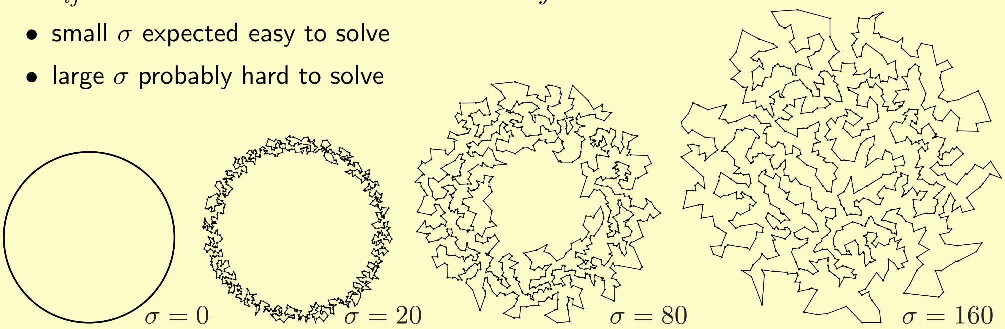
- very famous and well studied **NP**-hard problem
- important real world applications, e.g. vehicle routing



Here, the location of the cities is governed by the parameter σ.

- start with N cities on a circle with circumference $2\pi N$, i.e. the distance between neighboring cities ≈ 1
- move every city by random $\phi \in U[0, 2\pi)$ and $r \in U[0, \sigma]$
- c_{ij} is the Euclidean distance between i and j
- small σ expected easy to solve
- large σ probably hard to solve





The TSP as a LP [4]

3

Let x_{ij} be the adjacency matrix of the tour, i.e. 1 if i and j are consecutive in the tour, else 0. The following is known as the TSP LP relaxation.

minimize

$$\sum_i \sum_{j<i} c_{ij} x_{ij}$$

subject to

$$x_{ij} \in [0, 1]$$

$$\sum_j x_{ij} = 2 \qquad \forall i \in V$$

$$\sum_{i \in S, j \notin S} x_{ij} \geq 2 \qquad \forall S \subsetneq V, \; S \neq \varnothing$$

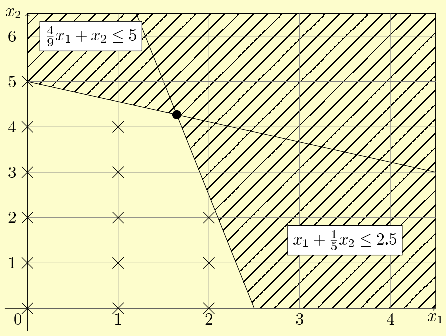
To get a full description the following integer constraint is also needed.

$$x_{ij} \in \{0, 1\}$$

Linear Programming (LP) [2]

2

- linear objective function to minimize/maximize
- linear constraints to define the feasible polytope, i.e. the solution space
- optimal solution always on the corners of the polytope
- solvable in polynomial-time, e.g. by *ellipsoid medthod*
- BUT integer constraints cannot be expressed by linear inequalities
- if solution is integer, it is an optimum of the integer problem
- otherwise this can be used as a starting point for *branch-and-bound* (respectively *branch-and-cut*)



- constraints can be added on demand by *cutting planes*
  - efficient solution of some problems with exponentially many constraints are possible [3]

Solution Probability p of the LP Relaxation

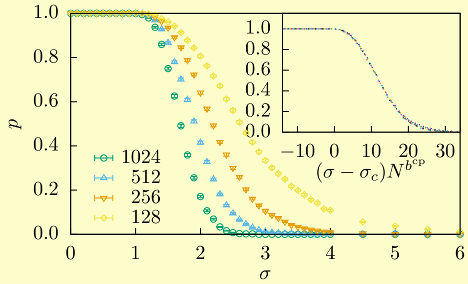
4

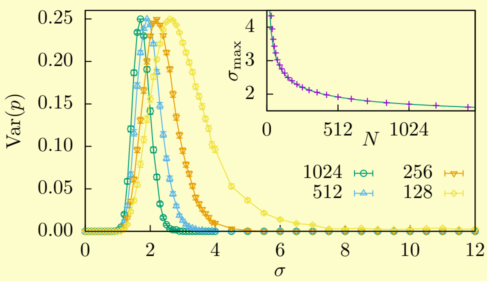
Phase transitions in optimization problems [6] are seldom examined with LP techniques, despite LP being ubiquitous in commercial optimization.

- transition from easy ($p = 1$) to hard ($p = 0$) (compare [5])
- at $\sigma_c = 1.08(7)$
- critical exponent $b = 0.43(4)$
- obtained by extrapolation of Variance peaks by fit to

$$\sigma_{\max} = aN^{-b} + \sigma_c$$

- data collapse works well
- other transitions are observable with more cutting planes, e.g. *Comb constraints*





Bibliography

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