CARL VON OSSIETZKY universität OLDENBURG

Linear Programming and Cutting Planes for Ground States and Excited States of the Traveling Salesperson Problem

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November 30, 2017

Traveling Salesperson Problem

Linear Programming

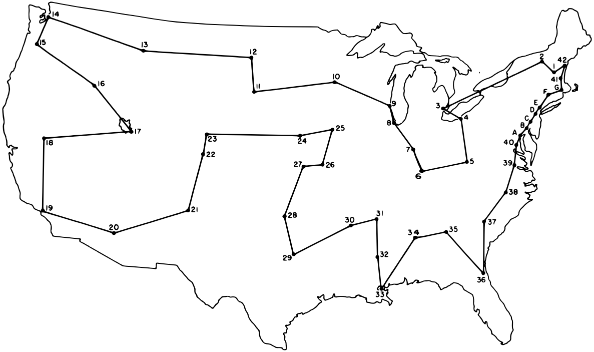
Integer Programming and Cutting Planes

Easy-Hard Transition

Exploring the Energy Landscape

Traveling Salesperson Problem

Given a set of cities V and their pairwise distances c_{ij}, what is the shortest tour visiting all cities and returning to the start?



from Dantzig, Fulkerson, Johnson, Journal of the Operations Research Society of America, 1954, 42 cities

Traveling Salesperson Problem

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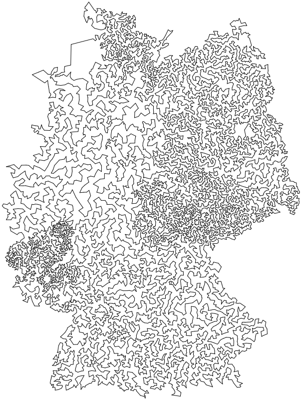

from Applegate, Bixby, Chvátal, Cook, 2001, 15112 cities

Traveling Salesperson Problem

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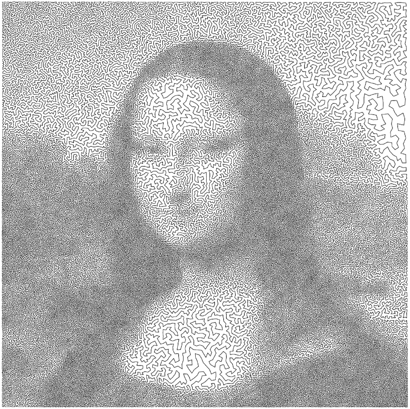


from Bosh, Herman, 2004, 100000 cities (not optimal, tour from 2009)

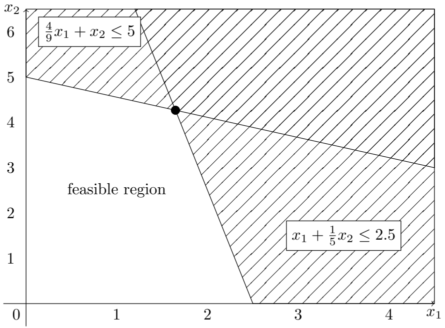
Linear Programming

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b}. \end{aligned}$$

$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \frac{4}{9} & 1 \\ 1 & \frac{1}{5} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 5 \\ 2.5 \end{pmatrix}$$



Linear Programming

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}. \end{aligned}$$

- polynomial time
- can be used for combinatorial (integer) problems
  - works outside the space of feasible solutions
  - is not always a valid solution
  - result valid $\Rightarrow$ result optimal
  - yields at least a lower bound

TSP as LP

let x_{ij} be the edge between cities i and j
$x_{ij} = 1$ if i and j are consecutive in the tour else 0
$c_{ij} = \mathrm{dist}(i, j)$ is the distance between city i and j

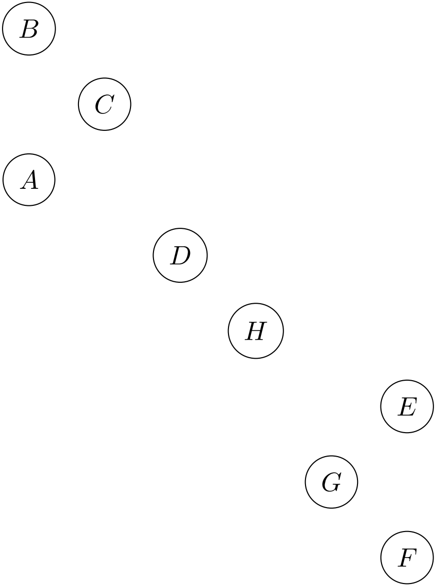
$$\text{minimize} \sum_i \sum_{j<i} c_{ij} x_{ij}$$

for example

$$x_{ij} = \begin{pmatrix} \cdot & 1 & 0 & 0 & 1 \\ 1 & \cdot & 0 & 1 & 0 \\ 0 & 0 & \cdot & 1 & 1 \\ 0 & 1 & 1 & \cdot & 0 \\ 1 & 0 & 1 & 0 & \cdot \end{pmatrix}$$

is the cyclic tour $(1, 2, 4, 3, 5)$

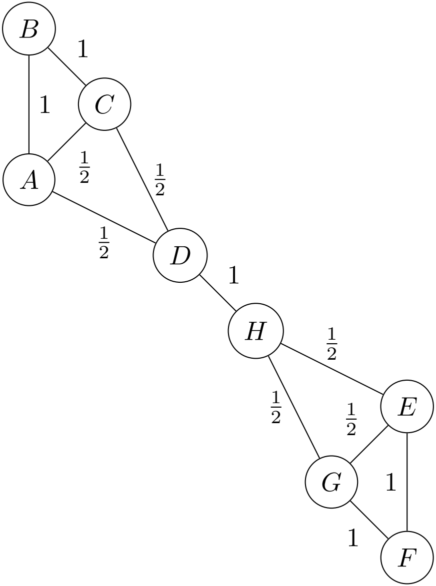
Constraints



Constraints

$$\sum_j x_{ij} = 2 \quad \forall i \in V$$

- every city needs 2 ways



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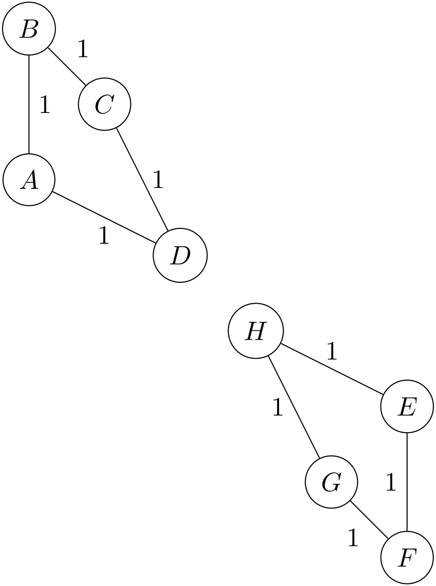


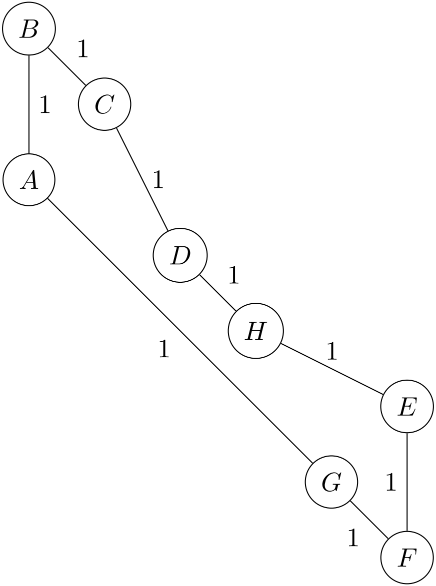
Constraints

$$\sum_j x_{ij} = 2 \quad \forall i \in V$$

- every city needs 2 ways

$$\sum_{i \in S, j \notin S} x_{ij} \geq 2 \quad \forall S \subset V$$

- kills subtours/loops
- kills some fractional solutions
- global min-cut to find



Constraints

$$\text{minimize} \quad \sum_i \sum_{j<i} c_{ij} x_{ij}$$

$$\text{subject to} \quad x_{ij} \in \{0, 1\}$$

$$\sum_j x_{ij} = 2 \qquad i = 1, 2, ..., N \qquad \text{(IO)}$$

$$\sum_{i \in S, j \notin S} x_{ij} \geq 2 \qquad \forall S \subset V, S \neq \varnothing, S \neq V \qquad \text{(SEC)}$$

- x_{ij} are restricted to integer
  - relax/ignore this and cope with it later
- $\forall S \subset V$ are exponentially many
  - add only violated

Dantzig, Fulkerson, Johnson, J. Oper. Res. Soc. Am., 2 (1954) 393

Generating a solution from a LP relaxation

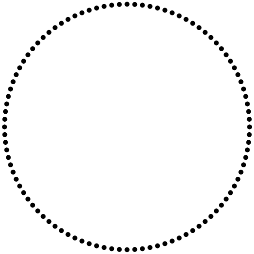
- more sophisticated cutting planes
  - Blossom inequalities
  - Comb inequalities
  - ...
- Branch-and-Bound or Branch-and-Cut
  - Combine with heuristics to lower the bound

e.g. implemented in Concorde (Applegate, Bixby, Chvátal, Cook)

Fuzzy Circle Ensemble (FCE)

Ensemble of disordered circles driven by the parameter σ

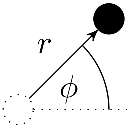
1. N cities on a circle
   with $R = N/2\pi$



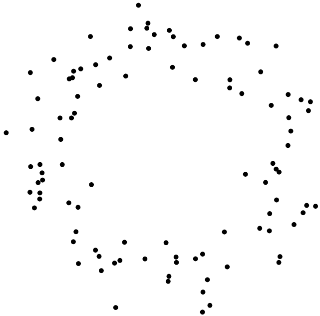
Fuzzy Circle Ensemble (FCE)

Ensemble of disordered circles driven by the parameter σ

1. N cities on a circle with $R = N/2\pi$
2. displace cities randomly



$r \in U[0, \sigma], \phi \in U[0, 2\pi)$



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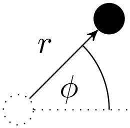
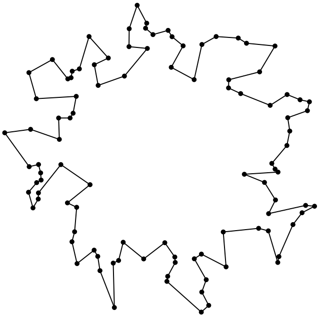


$r \in U[0, \sigma], \phi \in U[0, 2\pi)$

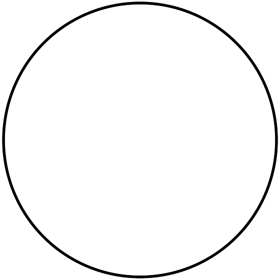
3. optimize the tour



Is there a phase transition easy circle $\rightarrow$ hard realization?

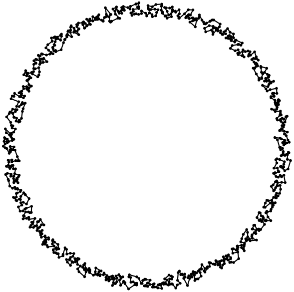
FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 0$



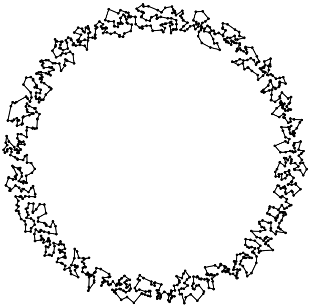
FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 10$



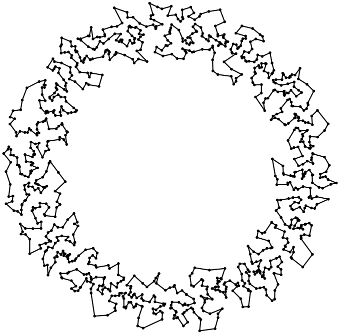
FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 20$



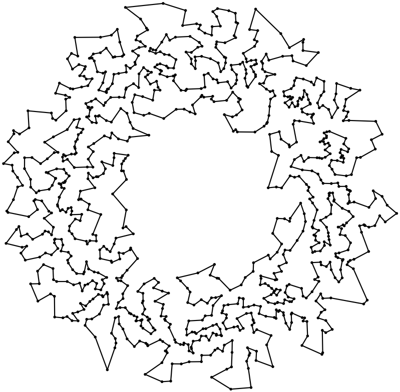
FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 40$



FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 80$



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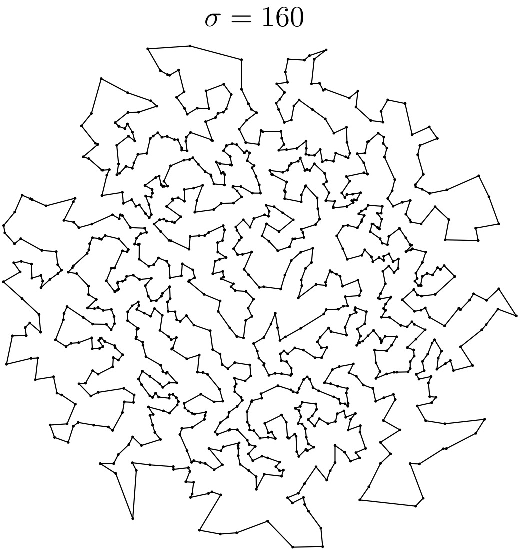
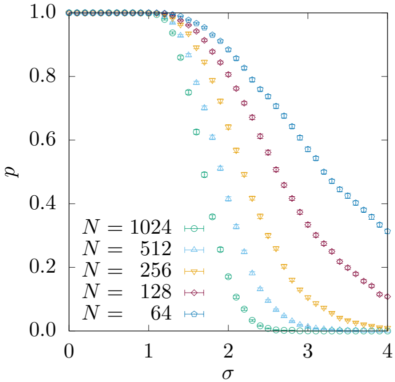
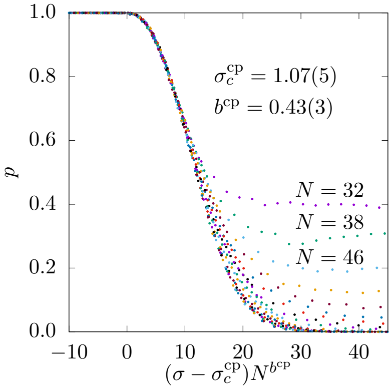


Solution probability p

Probability p that the SEC-relaxation is integer





Schawe, Hartmann, EPL **113** (2016) 30004

Structural Properties

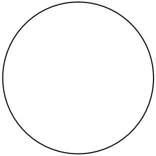
Observable is surely method dependent

search for "physical" properties of the optimal tours

- solve them by branch-and-cut
- do structural properties change at the transition points?

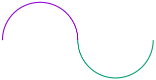
Tortuosity

$$\tau = \frac{n-1}{L} \sum_{i=1}^{n} \left(\frac{L_i}{S_i} - 1 \right)$$

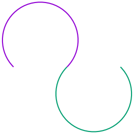


$\tau = 0$

$\tau = 0$



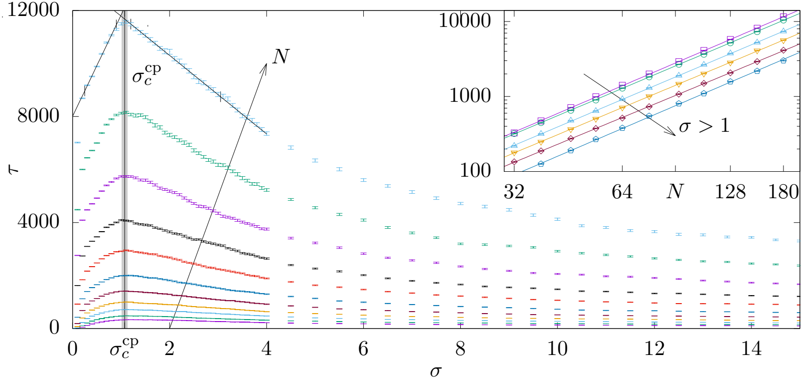
$\tau \approx 1.3$



$\tau \approx 4.1$

$\tau \approx 2.4$

Tortuosity



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Exploring the Energy Landscape (Work in Progress)

Complex Energy Landscape

change a fraction of an infinite system with finite energy

more precise

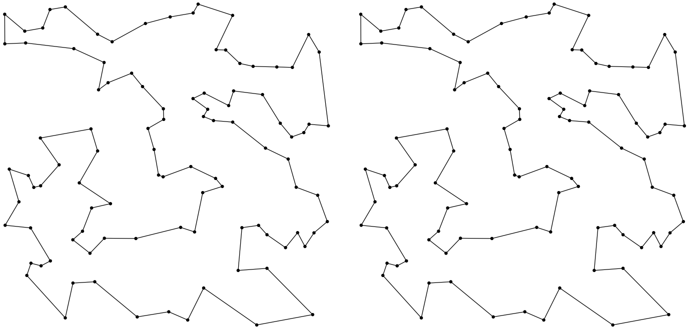
if relative difference of T^* and T^o in energy goes as $O(\frac{1}{N})$ and their difference goes as $O(N)$ $\Rightarrow$ sign of broken replica symmetry

Spinglass	TSP
Energy	Tour Length
Ground State	Optimal Tour
Link Overlap	Fraction of common Edges

Mézard and Parisi, J. Physique, **47** (1986) 1285-1296

Exotic Constraints

Optimal tour (T^o)

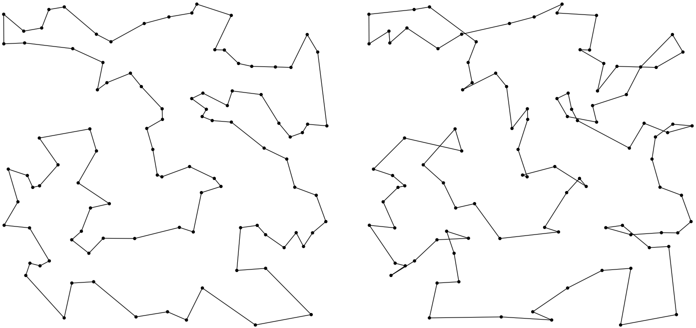


Exotic Constraints

Most different tour from optimum within some ϵ of length

$$\text{minimize} \sum_{\{i,j\} \in T^o} x_{ij}$$

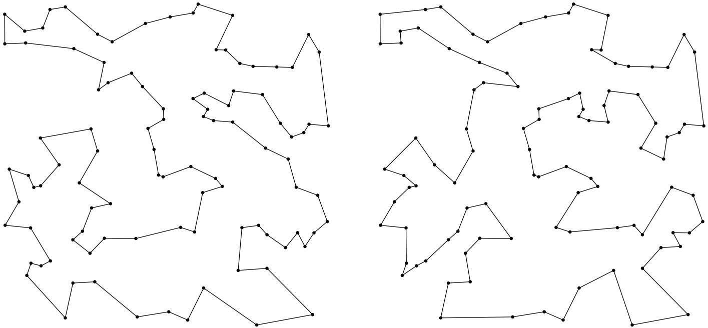
$$\sum_i \sum_{j<i} c_{ij} x_{ij} \leq L^o + \epsilon$$



Exotic Constraints

Add a penalty to the optimal edges

$$\text{minimize} \sum_i \sum_{j<i} c_{ij} x_{ij} + \frac{\epsilon}{N} \sum_{\{i,j\} \in T^o} x_{ij}$$



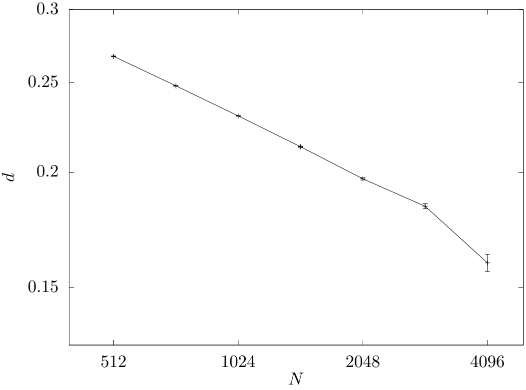
Preliminary Results

The Euclidean TSP energy landscape seems trivial

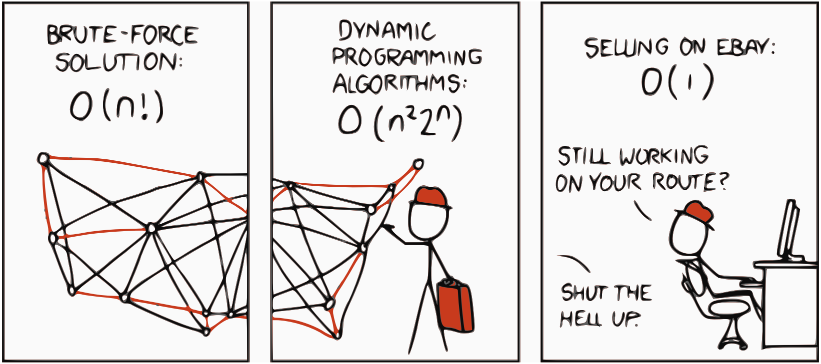
everything we tested decays with increasing system size

Hints that conjectured replica symmetry holds

before tested for uncorrelated distances



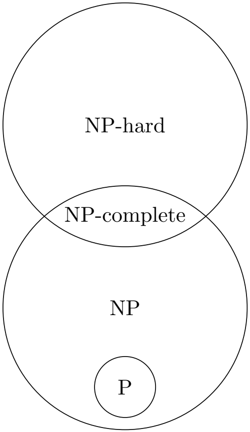
Thank you for listening



What's the complexity class of the best linear programming cutting-plane techniques? I couldn't find it anywhere. Man, the Garfield guy doesn't have these problems ...

NP{,-complete,-hard}

- P
  - decision problem
  - solvable in polynomial-time
  - e.g. "Is x prime?"
- NP
  - decision problem
  - verifiable in polynomial-time
  - e.g. "Is x composite?"
- NP-hard
  - any problem in NP can be reduced to one in NP-hard
  - e.g. TSP, Spinglass Groundstates
- NP-complete
  - is the intersection of NP and NP-hard
  - e.g. SAT, Vertex Cover, TSP-decision



if P $\neq$ NP

Tortuosity

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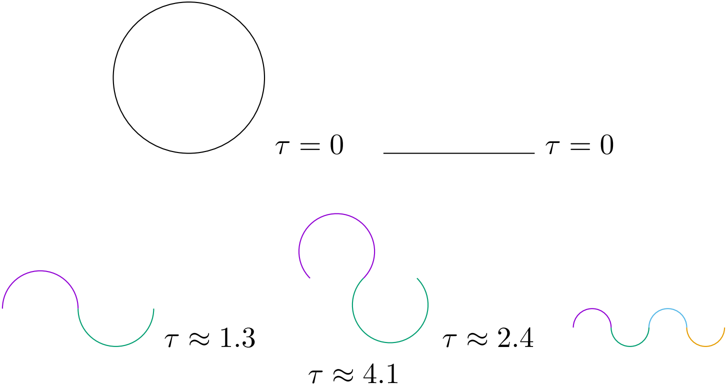

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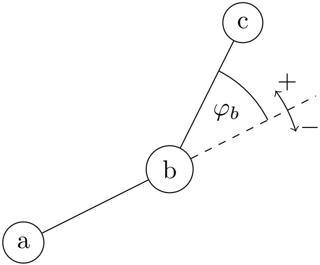

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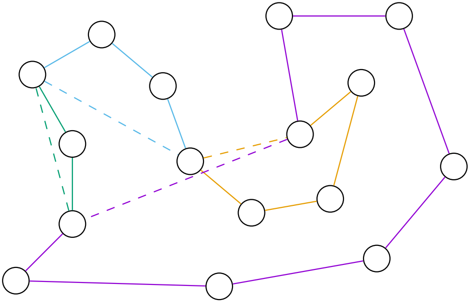

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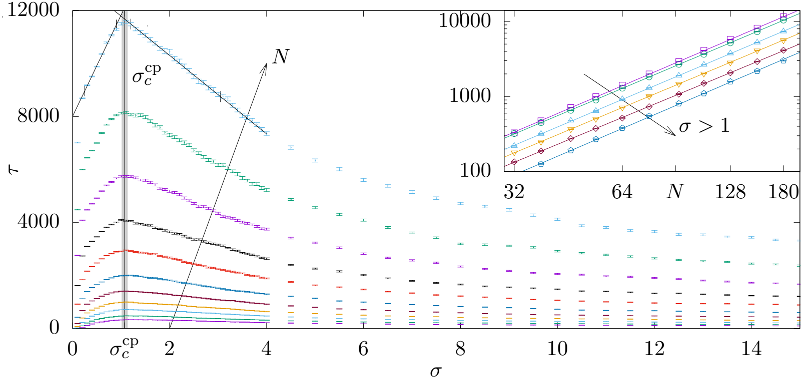

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Stör-Wagner Global Minimum Cut[7]

- $\mathcal{O}(|V||E| + |V|^2 \log |V|)$

1. find an arbitrary s-t-min-cut
2. merge s and t
3. repeat until one vertex is left
4. smallest encountered s-t-min-cut is global min-cut

[7] M. Stör and F. Wagner, JACM, 1997

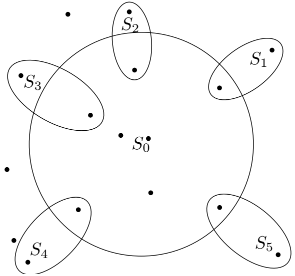
Blossom Inequalities

$$\sum_{m=0}^{k} \sum_{i \in S_m, j \notin S_m} x_{ij} \geq 3k+1$$

$$\begin{aligned} k \text{ odd} & \\ S_i \cap S_j = \varnothing & \qquad \forall i, j \in \{1, \dots, k\} \\ S_0 \cap S_i \neq \varnothing & \qquad \forall i \in \{1, \dots, k\} \\ S_i \setminus S_0 \neq \varnothing & \qquad \forall i \in \{1, \dots, k\} \\ |S_i| = 2 & \qquad \forall i \in \{1, \dots, k\} \end{aligned}$$

Blossom Inequalities

$$\sum_{m=0}^{k} \sum_{i \in S_m, j \notin S_m} x_{ij} \geq 3k + 1$$



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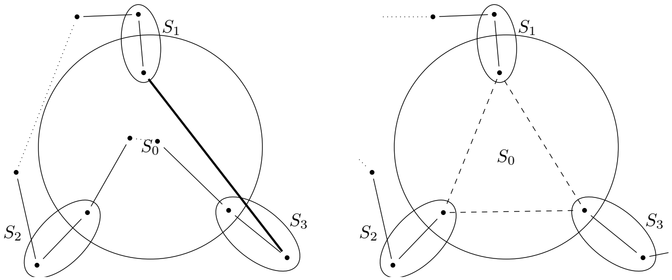
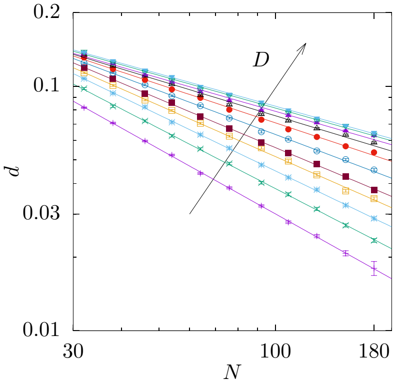
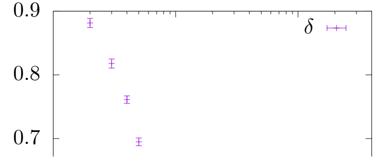


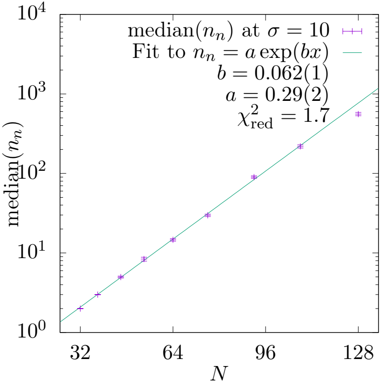
First Excitation: The Second Shortest Tour

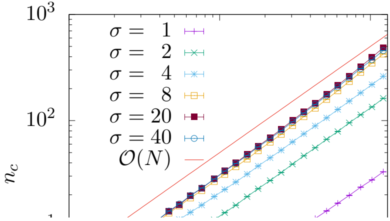
Uniformly distributed cities in high dimensions $2 \leq D \leq 312$.





Runtime Measurements





Universality

Same analysis with other ensembles (Gaussian displacement, displacement in three dimensions, some blossom inequalities)

	σ_c	b
Degree relaxation	$\sigma_c^{\mathrm{lp}} = 0.51(4)$	$b^{\mathrm{lp}} = 0.29(6)$
SEC relaxation	$\sigma_c^{\mathrm{cp}} = 1.07(5)$	$b^{\mathrm{cp}} = 0.43(3)$
	$\sigma_c^{\tau} = 1.06(23)$	–
	$\sigma_c^{\mathrm{cp,g}} = 0.47(3)$	$b^{\mathrm{cp,g}} = 0.45(5)$
	$\sigma_c^{\tau\mathrm{,g}} = 0.44(8)$	–
	$\sigma_c^{\mathrm{cp,3}} = 1.18(8)$	$b^{\mathrm{cp,3}} = 0.40(4)$
fast Blossom rel.	$\sigma_c^{\mathrm{fb}} = 1.47(8)$	$b^{\mathrm{fb}} = 0.40(3)$