Linear Programming and Cutting Planes for Ground States and Excited States of the Traveling Salesperson Problem

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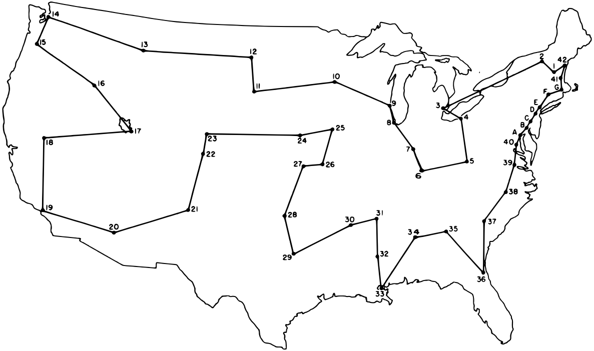
March 12, 2018

Traveling Salesperson Problem

Given a set of cities V and their pairwise distances c_{ij}, what is the shortest tour visiting all cities and returning to the start?



Dantzig, Fulkerson, Johnson, Journal of the Operations Research Society of America, 1954, 42 cities

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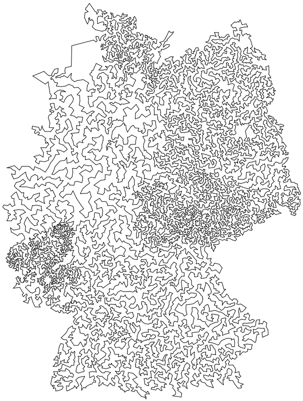

Applegate, Bixby, Chvátal, Cook, 2001, 15112 cities

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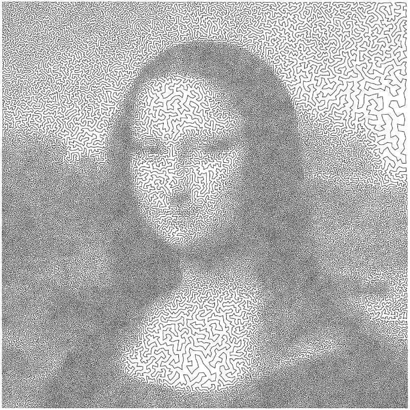
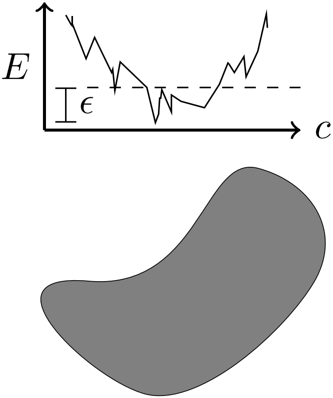


Bosh, Herman, 2004, 100000 cities (not optimal, tour from 2009)

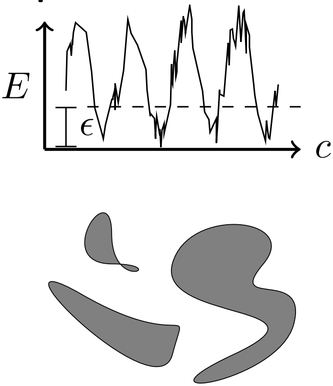
What does the energy landscape look like?

trivial



Random Field Ising
3-SAT ($\alpha < 3.86$)

complex



SK Spinglasses
3-SAT ($\alpha > 3.86$)

TSP?

Test for Complex Energy Landscape

For $N \to \infty$, can a **finite** increase in energy change a **finite fraction** of the system?

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Traveling Salesperson Problem

- compare two realizations
  - T^o the optimal tour
  - T^* some excitation
- such that their relative length difference $\frac{L^o - L^*}{L^o} \sim \mathcal{O}(1/N)$
- does their difference grows as $d \sim \mathcal{O}(N) \Leftrightarrow \frac{d}{N} \sim \mathcal{O}(1)$

Mézard and Parisi, J. Physique, **47** (1986) 1285-1296

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We just need to generate excitations!

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We just need to generate excitations!
Oh, and the optimal tour.

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TSP as a Linear Program

let x_{ij} be the edge between cities i and j
$x_{ij} = 1$ if i and j are consecutive in the tour else 0
$c_{ij} = \mathrm{dist}(i, j)$ is the distance between city i and j

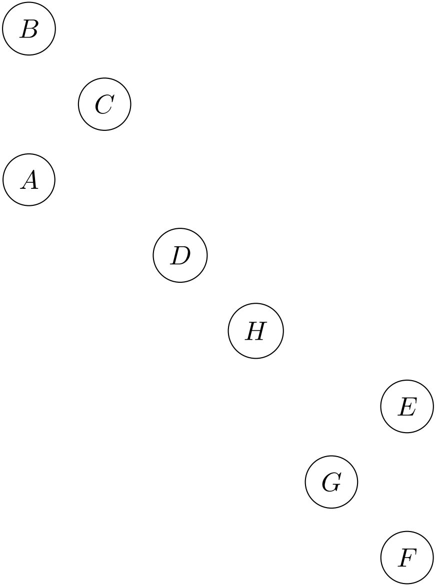
$$\text{minimize} \sum_i \sum_{j<i} c_{ij} x_{ij}$$

for example

$$x_{ij} = \begin{pmatrix} \cdot & 1 & 0 & 0 & 1 \\ 1 & \cdot & 0 & 1 & 0 \\ 0 & 0 & \cdot & 1 & 1 \\ 0 & 1 & 1 & \cdot & 0 \\ 1 & 0 & 1 & 0 & \cdot \end{pmatrix}$$

is the cyclic tour $(1, 2, 4, 3, 5)$

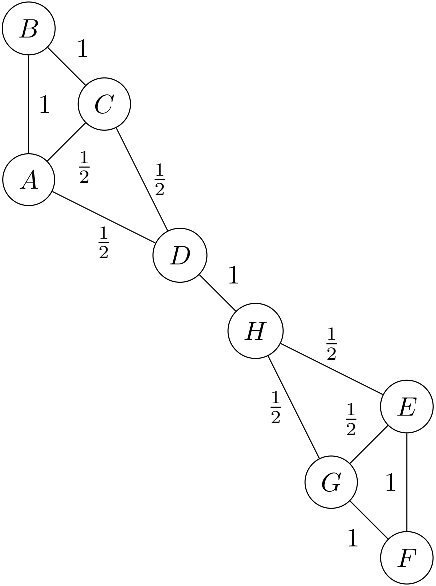
Constraints



Constraints

$$\sum_j x_{ij} = 2 \quad \forall i \in V$$

- every city needs 2 ways



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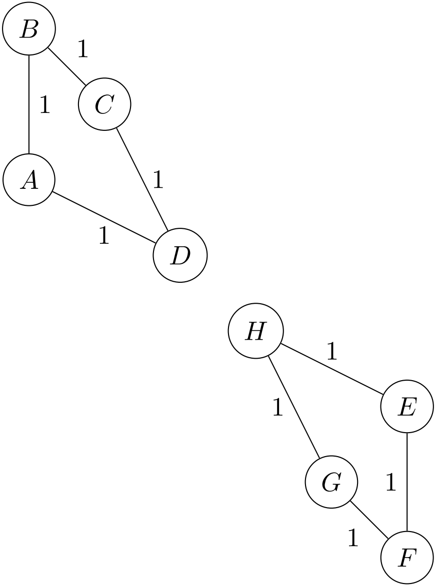


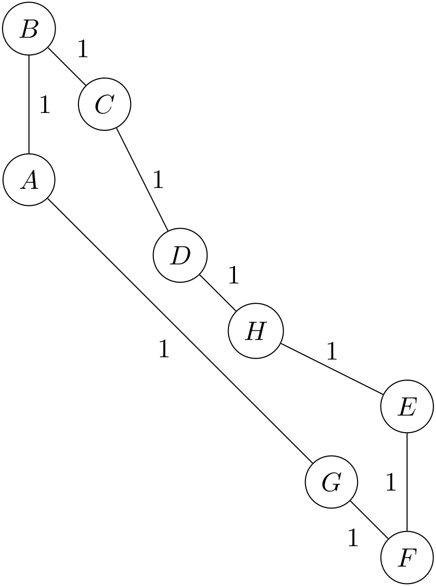
Constraints

$$\sum_j x_{ij} = 2 \quad \forall i \in V$$

- every city needs 2 ways

$$\sum_{i \in S, j \notin S} x_{ij} \geq 2 \quad \forall S \subset V$$

- kills subtours/loops
- kills some fractional solutions
- global min-cut to find



Optimal Tour

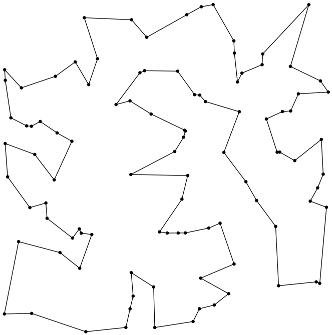
minimize $\displaystyle\sum_i \sum_{j<i} c_{ij} x_{ij}$

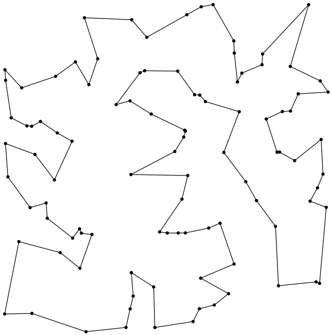
subject to

$$x_{ij} \in \{0,1\}$$

$$\sum_j x_{ij} = 2 \qquad i = 1, 2, ..., N$$

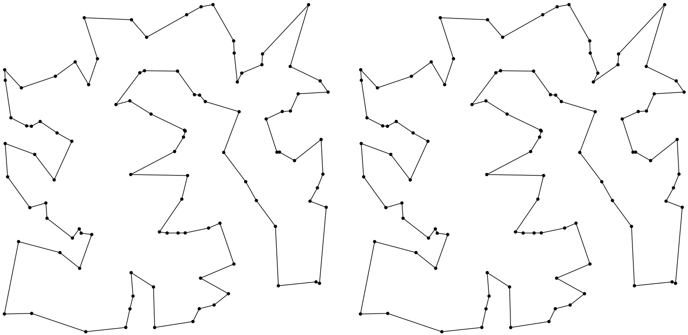
$$\sum_{i \in S, j \notin S} x_{ij} \geq 2 \qquad \forall S \subset V$$





Second Shortest Tour

$$\begin{aligned}
\text{minimize} \quad & \sum_i \sum_{j<i} c_{ij} x_{ij} \\
\text{subject to} \quad & x_{ij} \in \{0,1\} \\
& \sum_j x_{ij} = 2 \qquad i = 1, 2, ..., N \\
& \sum_{i \in S, j \notin S} x_{ij} \geq 2 \qquad \forall S \subset V \\
& \color{red}{\sum_{\{i,j\} \in T^o} x_{ij} < N}
\end{aligned}$$



Second Shortest Tour

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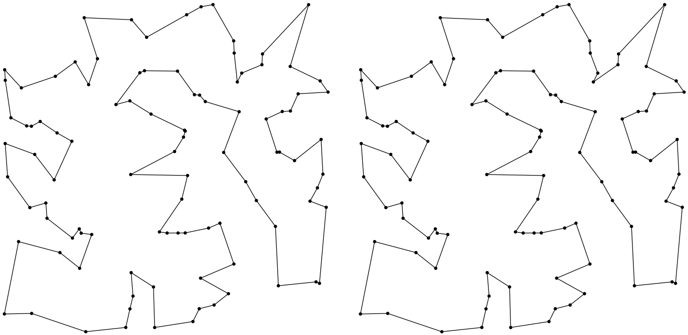
$$

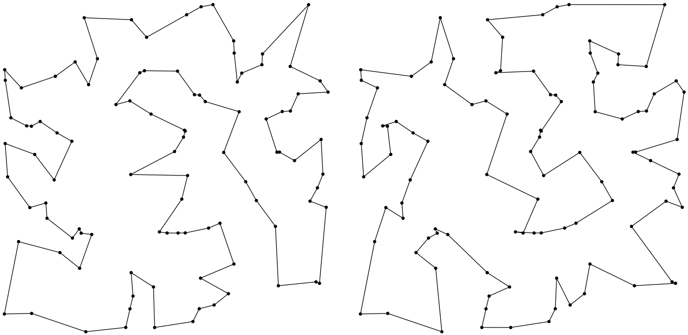
For $N \to \infty$, can a **finite** increase in energy change a **finite fraction** of the system?



Shortest Half Different

$$\begin{aligned}
\text{minimize} \quad & \sum_i \sum_{j<i} c_{ij} x_{ij} \\
\text{subject to} \quad & x_{ij} \in \{0,1\} \\
& \sum_j x_{ij} = 2 \qquad i = 1,2,...,N \\
& \sum_{i \in S, j \notin S} x_{ij} \geq 2 \qquad \forall S \subset V \\
& \sum_{\{i,j\} \in T^o} x_{ij} \leq N/2
\end{aligned}$$

For $N \to \infty$, can a **finite** increase in energy change a **finite fraction** of the system?

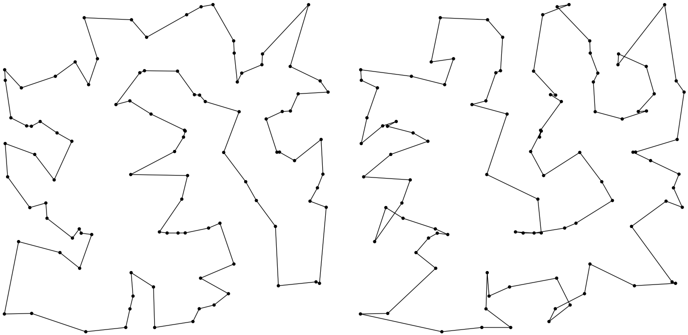


Schawe, Jha, Hartmann

Most Different Short Tour

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{\{i,j\}\in T^o} x_{ij} & \\
\text{subject to} & x_{ij} \in \{0,1\} & \\
 & \displaystyle\sum_{j} x_{ij} = 2 & i = 1,2,...,N \\
 & \displaystyle\sum_{i\in S, j\notin S} x_{ij} \geq 2 & \forall S \subset V \\
 & \displaystyle\sum_{i}\sum_{j<i} c_{ij}x_{ij} \leq L^o + \epsilon &
\end{array}
$$

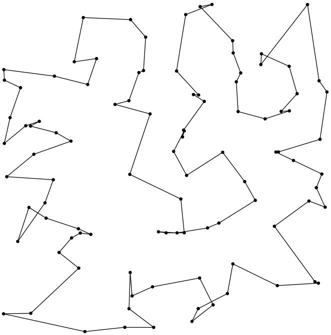
For $N \to \infty$, can a **finite** increase in energy change a **finite fraction** of the system?

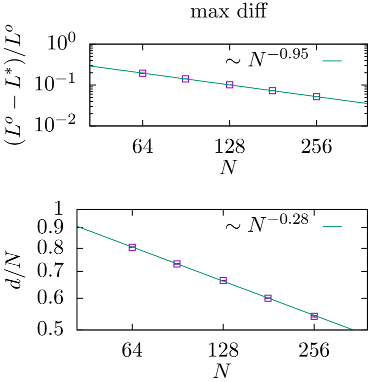


Results

For $N \to \infty$, can a **finite** increase in energy change a **finite fraction** of the system?

Euclidean TSP in plane is **trivial**





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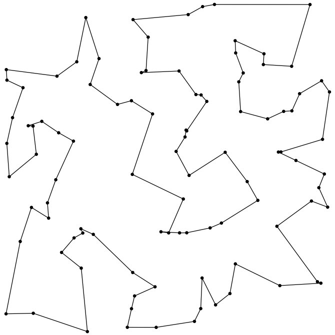

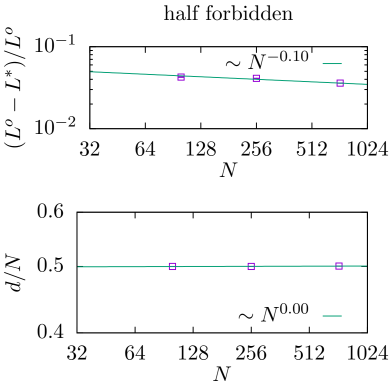

Results

Disturbed Square Lattice ($\sigma \sim \frac{1}{N}$) is **complex**

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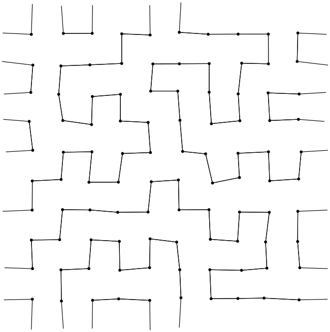

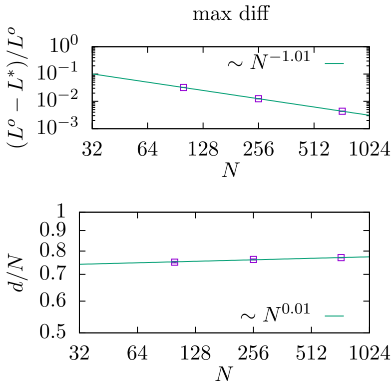

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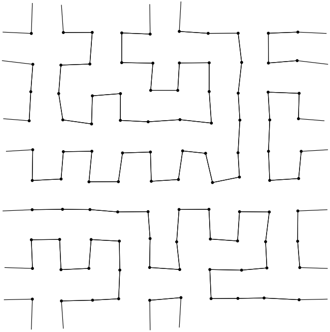
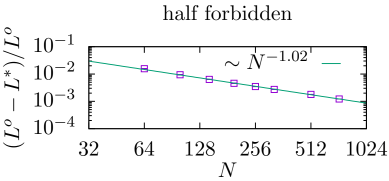
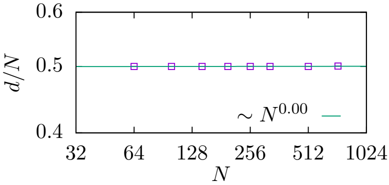
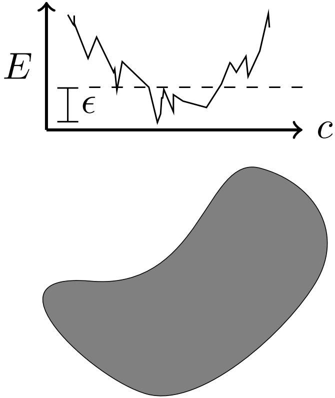






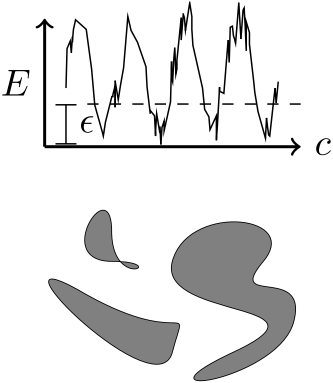
This is how the energy landscape looks like

trivial



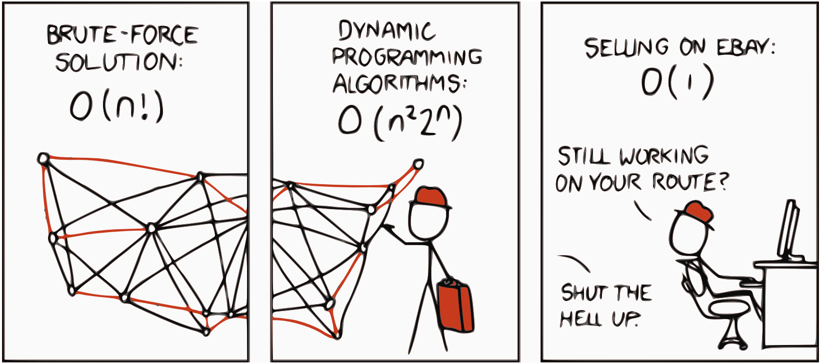
Euclidean TSP
Diluted TSP
"SK TSP"

complex



Disturbed Square Lattice TSP

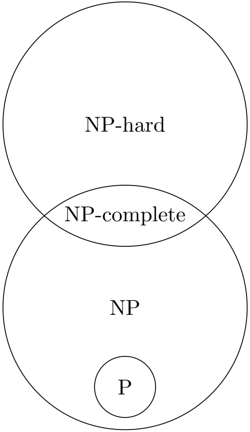
Thank you for listening



What's the complexity class of the best linear programming cutting-plane techniques? I couldn't find it anywhere. Man, the Garfield guy doesn't have these problems ...

NP{,-complete,-hard}

- P
  - decision problem
  - solvable in polynomial-time
  - e.g. “Is x prime?”
- NP
  - decision problem
  - verifiable in polynomial-time
  - e.g. “Is x composite?”
- NP-hard
  - any problem in NP can be reduced to one in NP-hard
  - e.g. TSP, Spinglass Groundstates
- NP-complete
  - is the intersection of NP and NP-hard
  - e.g. SAT, Vertex Cover, TSP-decision



if P $\neq$ NP

Stör-Wagner Global Minimum Cut[7]

- $\mathcal{O}(|V||E| + |V|^2 \log |V|)$

1. find an arbitrary s-t-min-cut
2. merge s and t
3. repeat until one vertex is left
4. smallest encountered s-t-min-cut is global min-cut

[7] M. Stör and F. Wagner, JACM, 1997

Blossom Inequalities

$$\sum_{m=0}^{k} \sum_{i \in S_m, j \notin S_m} x_{ij} \geq 3k + 1$$

$$\begin{aligned}
k \text{ odd} & \\
S_i \cap S_j = \varnothing & \qquad \forall i, j \in \{1, \dots, k\} \\
S_0 \cap S_i \neq \varnothing & \qquad \forall i \in \{1, \dots, k\} \\
S_i \setminus S_0 \neq \varnothing & \qquad \forall i \in \{1, \dots, k\} \\
|S_i| = 2 & \qquad \forall i \in \{1, \dots, k\}
\end{aligned}$$

Blossom Inequalities

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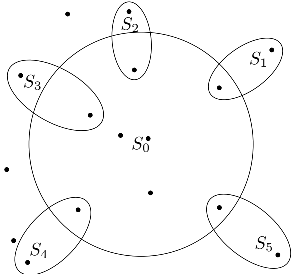

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