CARL VON OSSIETZKY universität OLDENBURG

Linear Programming by Example: The Travelling Salesperson Problem

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November 11, 2016

Travelling Salesperson Problem

Simple Heuristics

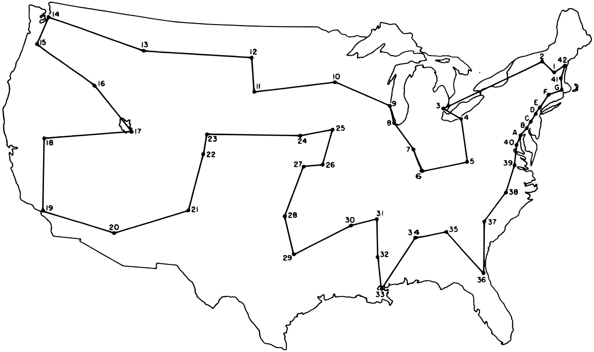
Linear Programming

Integer Programming and Cutting Planes

Current Research

Travelling Salesperson Problem

Given a set of cities V and their pairwise distances c_{ij}, what is the shortest tour visiting all cities and returning to the start?



from Dantzig, Fulkerson, Johnson, Journal of the Operations Reasearch Society of America, 1954, 42 cities

Travelling Salesperson Problem

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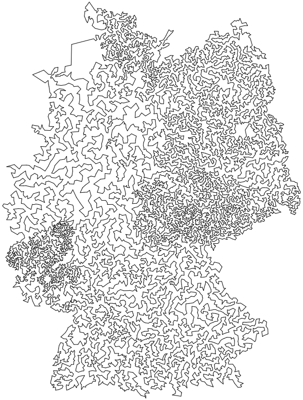

from Applegate, Bixby, Chvátal, Cook, 2001, 15112 cities

Travelling Salesperson Problem

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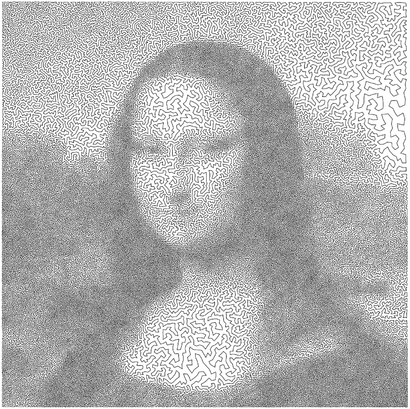

from Bosh, Herman, 2004, 100000 cities (not optimal, tour from 2009)

Travelling Salesperson Problem

- symmetric TSP
  - $c_{ab} = c_{ba}$
- metric TSP
  - triangle inequality $c_{ab} + c_{bc} \geq c_{ac}$
- Euclidean TSP
  - c_{ij} from Euclidean metric
  - polynomial-time approximation scheme (PTAS) exists[1]
  - but still NP-hard[2]
- Used for
  - vehicles
  - circuit board drills
  - testbed for metaheuristics (Simulated Annealing[3], Taboo Search[4], Ant Colony[5] and more)

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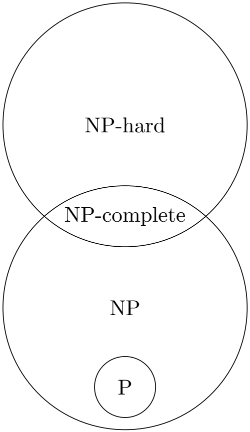
[1] Arora, JACM, 1998 [2] Papadimitriou, Theo. Comp. Science, 1977 [3] Kirkpatrick et al., science, 1983 [4] Glover, Comp. and OR, 1986 [5] Dorigo et al., Evo Comp IEEE, 1997

NP{,-complete,-hard}

- P
  - decision problem
  - solvable in polynomial-time
  - e.g. "Is x prime?"
- NP
  - decision problem
  - verifiable in polynomial-time
  - e.g. "Is x composite?"
- NP-hard
  - any problem in NP can be reduced to one in NP-hard
  - e.g. TSP, Spinglass Groundstates
- NP-complete
  - is the intersection of NP and NP-hard
  - e.g. SAT, Vertex Cover, TSP-decision



if P $\neq$ NP

Simple Heuristics

- Nearest Neighbor Heurist
- Greedy Heuristic
- Insertion Heuristics
- k-Opt Heuristics

$\rightarrow$ Visualization: `tspView`

Simple Heuristics

- Nearest Neighbor Heurist
- Greedy Heuristic
- Insertion Heuristics
- k-Opt Heuristics

How to judge if a solution is good, i.e., near the optimum?

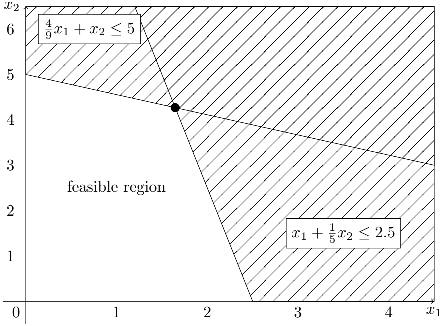
Linear Programming

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}. \end{aligned}$$

$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \frac{4}{9} & 1 \\ 1 & \frac{1}{5} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 5 \\ 2.5 \end{pmatrix}$$



Linear Programming

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}. \end{aligned}$$

- works outside the space of feasible solutions
- polynomial time
- can be used for combinatorial (integer) problems
  - is not always a valid solution
  - result valid $\rightarrow$ result optimal
  - yields at least a lower bound

TSP as LP

let x_{ij} be the edge between cities i and j
$x_{ij} = 1$ if i and j are consecutive in the tour else 0
$c_{ij} = \mathrm{dist}(i, j)$ is the distance between city i and j

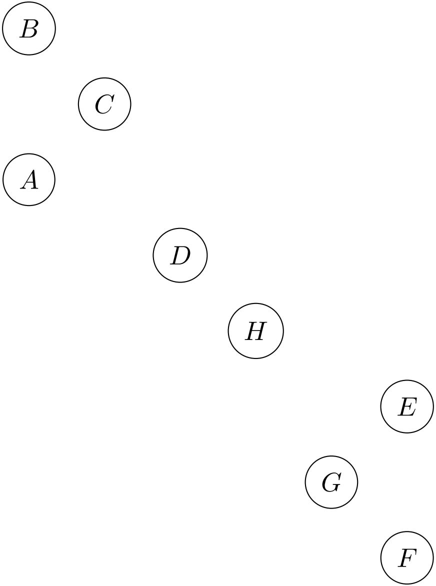
$$\text{minimize} \sum_i \sum_{j<i} c_{ij} x_{ij}$$

for example

$$x_{ij} = \begin{pmatrix} \cdot & 1 & 0 & 0 & 1 \\ 1 & \cdot & 0 & 1 & 0 \\ 0 & 0 & \cdot & 1 & 1 \\ 0 & 1 & 1 & \cdot & 0 \\ 1 & 0 & 1 & 0 & \cdot \end{pmatrix}$$

is the cyclic tour $(1, 2, 4, 3, 5)$

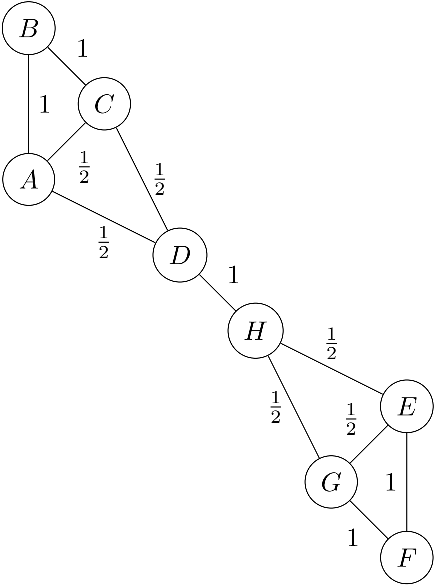
Constraints



Constraints

$$\sum_j x_{ij} = 2 \quad \forall i \in V$$

- every city needs 2 ways



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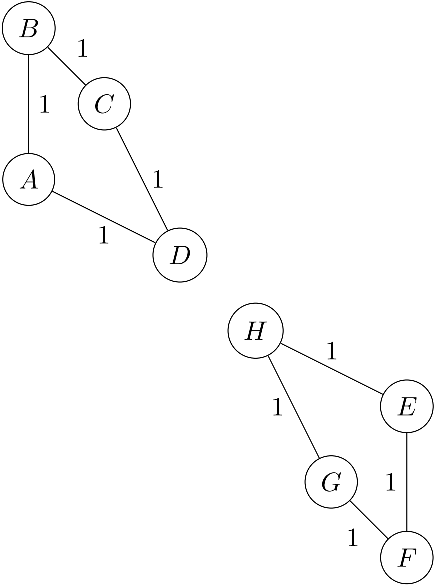


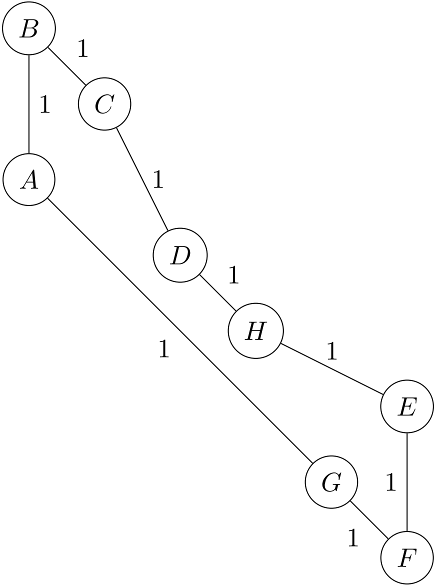
Constraints

$$\sum_{j} x_{ij} = 2 \quad \forall i \in V$$

- every city needs 2 ways

$$\sum_{i \in S, j \notin S} x_{ij} \geq 2 \quad \forall S \subset V$$

- kills subtours/loops
- kills some fractional solutions
- global min-cut to find



Constraints

$$
\begin{aligned}
\text{minimize} \quad & \sum_i \sum_{j<i} c_{ij} x_{ij} \\
\text{subject to} \quad & x_{ij} \in \mathbb{Z} \\
& \sum_j x_{ij} = 2 \quad i = 1, 2, ..., N \\
& \sum_{i \in S, j \notin S} x_{ij} \geq 2 \quad \forall S \subset V, S \neq \varnothing, S \neq V \quad \text{(SEC)}
\end{aligned}
$$

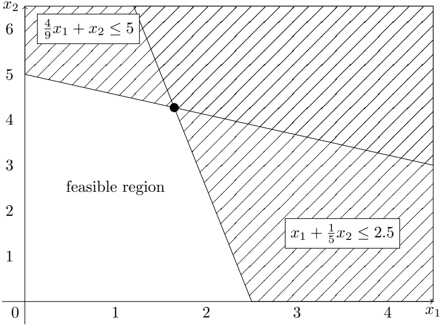
- x_{ij} are restricted to integer
  - relax/ignore this and cope with it later
- $\forall S \subset V$ are exponentially many
  - add only violated

Dantzig, Fulkerson, Johnson, J. Oper. Res. Soc. Am., 2 (1954) 393

Integer Programming and Cutting Planes

maximize $\quad x_1 + x_2$

$$x_i \in \mathbb{R}$$

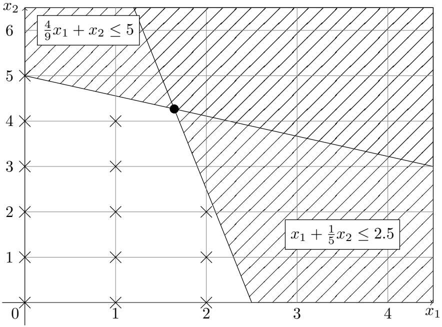


Integer Programming and Cutting Planes

maximize $\quad x_1 + x_2$

$$x_i \in \mathbb{Z}$$

- Branch and Bound
  - "clever backtracking"
- Cutting Planes
  - add constraint
  - invalidate relaxed optimum
  - all feasible solutions stay feasible



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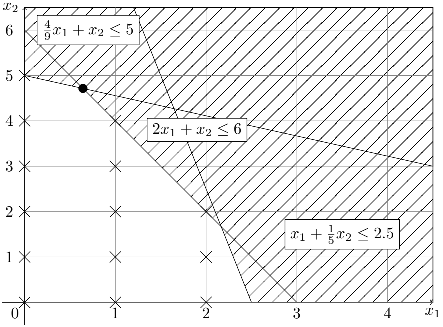

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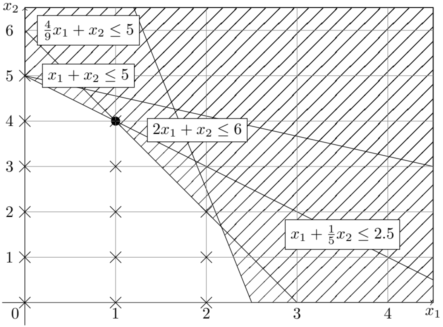


Visualization of some Examples

$\rightarrow$ Visualization: `tspView`

Generating a solution from a LP relaxation

- more sophisticated cutting planes
  - Blossom inequalities
  - Comb inequalities
  - ...
- Branch-and-Bound or Branch-and-Cut

Current Research in Statistical Physics

Phase transitions in optimization problems

Are there solvable $\rightarrow$ not solvable transitions?

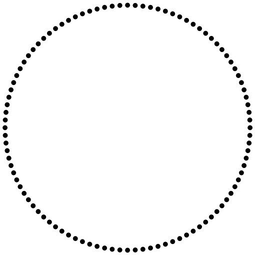
Are there easy $\rightarrow$ hard transitions?

LP-relaxation is integer $\rightarrow$ obtainable in polynomial-time $\rightarrow$ easy

Tunable Ensemble

Ensemble of disordered circles driven by the parameter σ

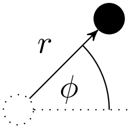
1. N cities on a circle
   with $R = N/2\pi$



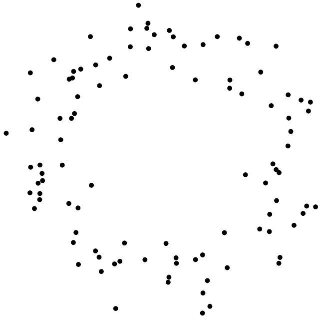
Tunable Ensemble

Ensemble of disordered circles driven by the parameter σ

1. N cities on a circle with $R = N/2\pi$
2. displace cities randomly



$r \in U[0, \sigma], \phi \in U[0, 2\pi)$



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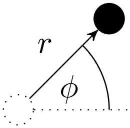
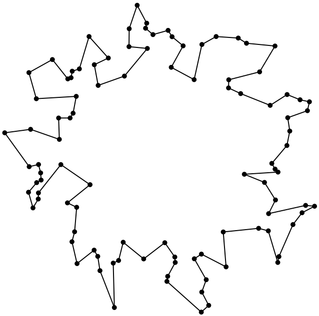


$r \in U[0,\sigma], \phi \in U[0,2\pi)$

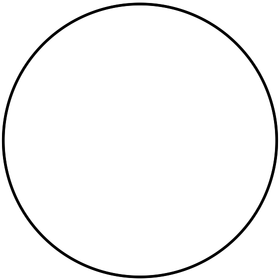
3. optimize the tour



Is there a phase transition easy circle $\rightarrow$ hard realization?

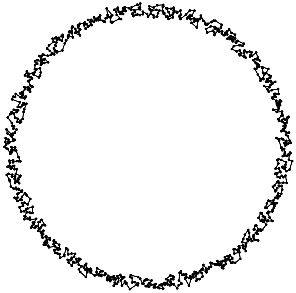
FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 0$



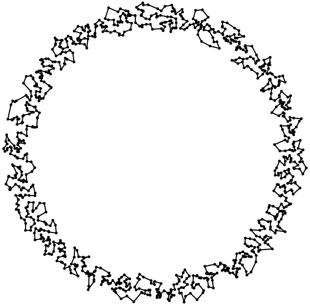
FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 10$



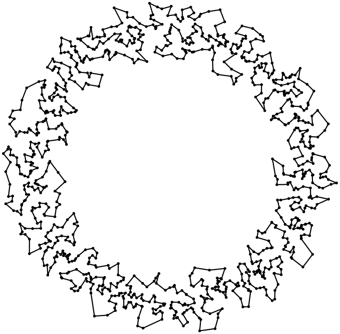
FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 20$



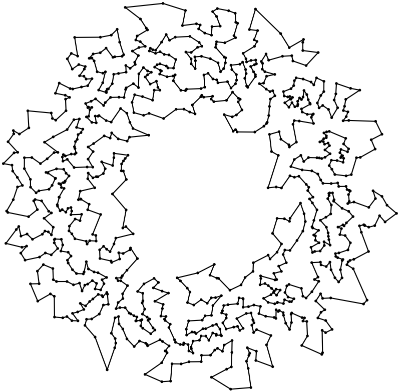
FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 40$



FCE Examples, $N = 1024, R = 1024/2\pi \approx 160$

$\sigma = 80$



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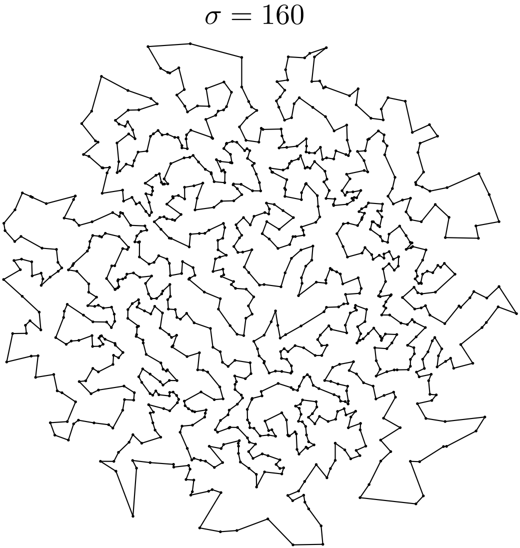
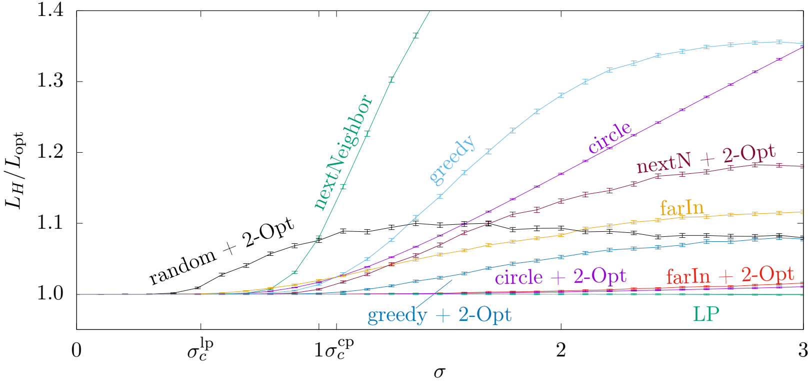


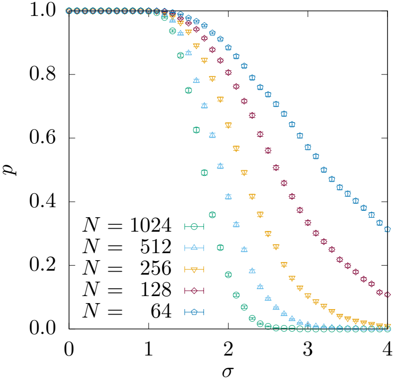
Performance of the heuristics

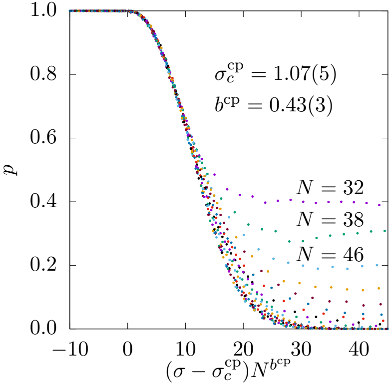
- easy for small σ
- LP gives a very tight lower bound



Solution probability p

Probability p that the SEC-relaxation is integer





Schawe, Hartmann, EPL **113** (2016) 30004

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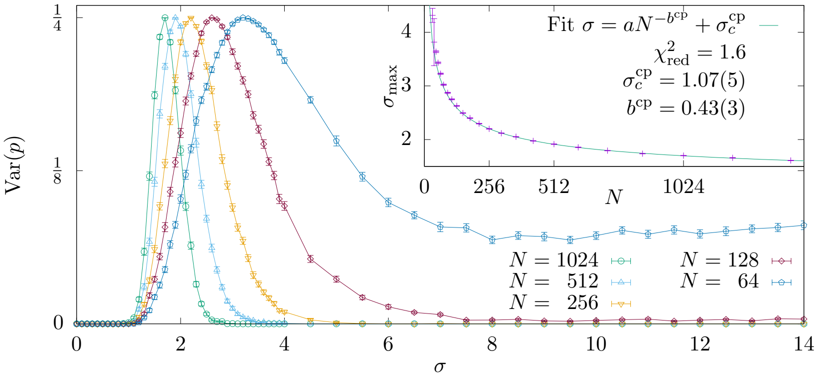

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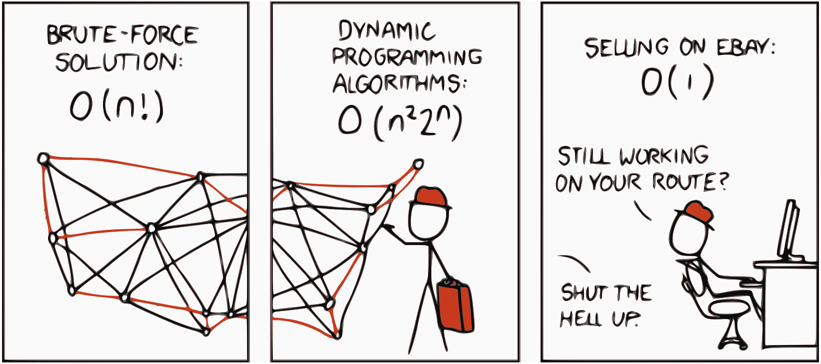
Take-home message

If you have a optimization problem and need to know the quality of your heuristic solution:

$\Rightarrow$ Consider linear programming.

Thank you for listening



What's the complexity class of the best linear programming cutting-plane techniques? I couldn't find it anywhere. Man, the Garfield guy doesn't have these problems ...

Stör-Wagner Global Minimum Cut[7]

- $\mathcal{O}(|V||E| + |V|^2 \log |V|)$

1. find an arbitrary s-t-min-cut
2. merge s and t
3. repeat until one vertex is left
4. smallest encountered s-t-min-cut is global min-cut

[7] M. Stör and F. Wagner, JACM, 1997

Blossom Inequalities

$$\sum_{m=0}^{k} \sum_{i \in S_m, j \notin S_m} x_{ij} \geq 3k + 1$$

$$\begin{aligned} k \text{ odd} & \\ S_i \cap S_j = \varnothing & \qquad \forall i, j \in \{1, \dots, k\} \\ S_0 \cap S_i \neq \varnothing & \qquad \forall i \in \{1, \dots, k\} \\ S_i \setminus S_0 \neq \varnothing & \qquad \forall i \in \{1, \dots, k\} \\ |S_i| = 2 & \qquad \forall i \in \{1, \dots, k\} \end{aligned}$$

Blossom Inequalities

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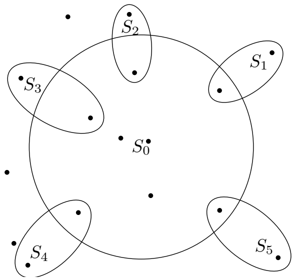

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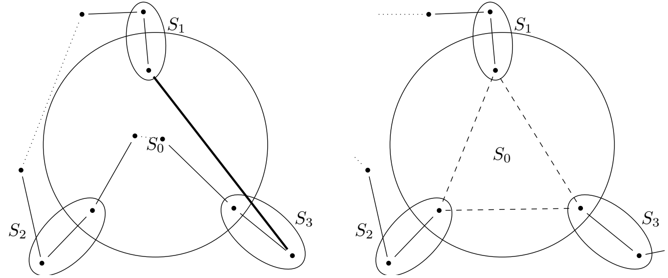
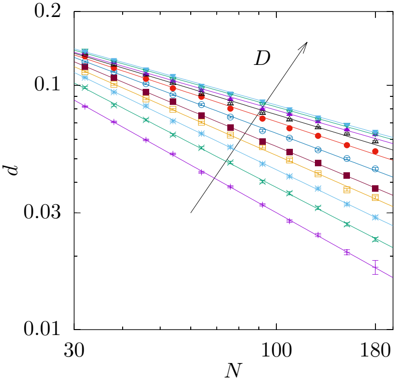
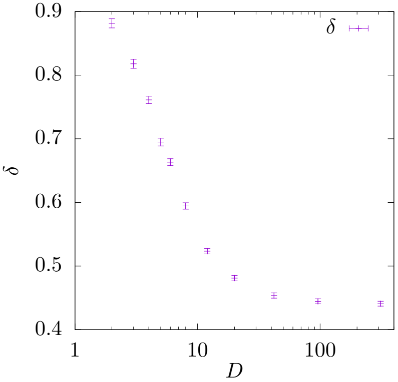


First Excitation: The Second Shortest Tour

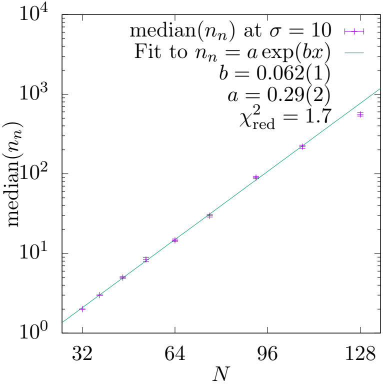
Uniformly distributed cities in high dimensions $2 \leq D \leq 312$.

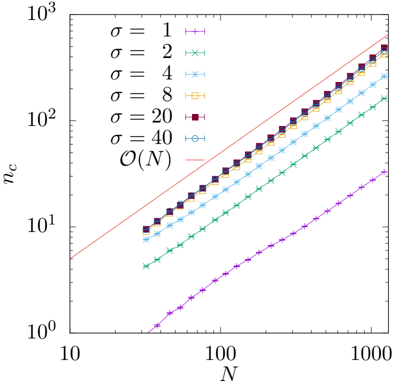




tour difference fitted to $d = aN^{-\delta}$

Runtime Measurements





Structural Properties

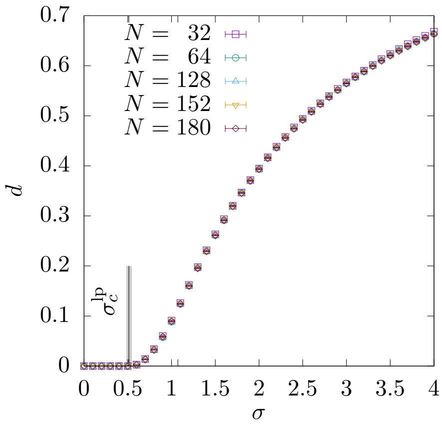
Those measurements are surely method dependent.
$\rightarrow$ search for "physical" properties of the optimal tours

- solve them by branch-and-cut
  - only possible for fairly small instances
- do structural properties change at the transition points?

Tour Difference d

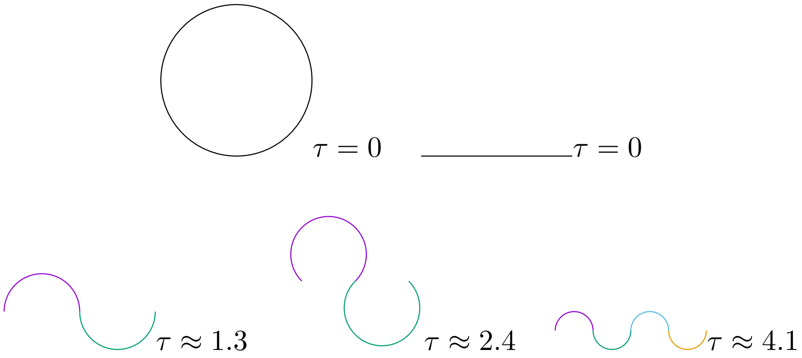
- Distance of two tours.
- Number of edges in the first tour, but not in the second.
  - $\sim$ Hamming Distance
- normalized by N

On the right: $d(x^{\circ}_{ij}, x^{*}_{ij})$, i.e. difference between inital circle and optimal tour.



Tortuosity

$$\tau = \frac{n-1}{L} \sum_{i=1}^{n} \left(\frac{L_i}{S_i} - 1 \right)$$



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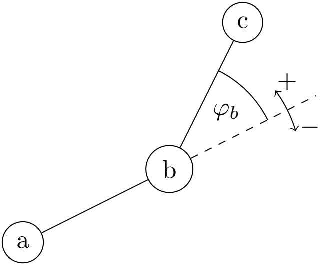

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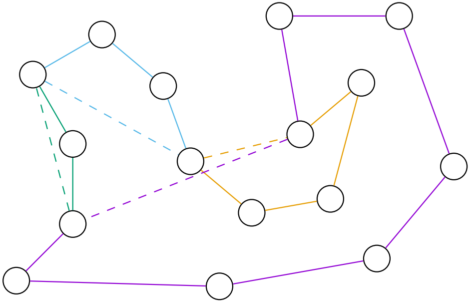

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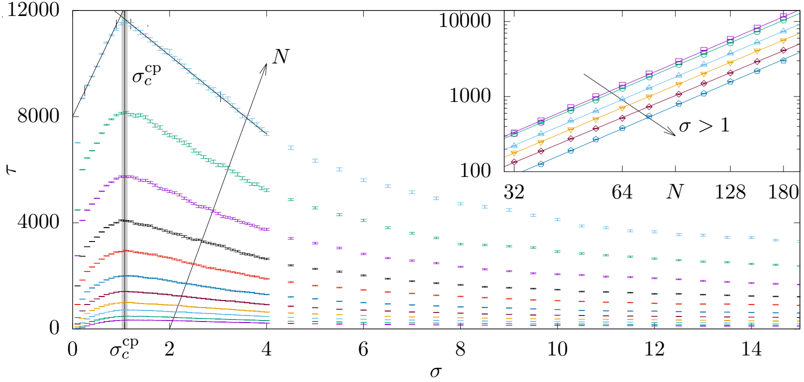

12000
8000
4000
0
τ
σ_c^{cp}
N
0
2
4
6
8
10
12
14
σ
10000
1000
100
32
64
128
180
$\sigma > 1$

Universality

Same analysis with other ensembles (Gaussian displacement, displacement in three dimensions, some blossom inequalities)

	σ_c	b
Degree relaxation	$\sigma_c^{\mathrm{lp}} = 0.51(4)$	$b^{\mathrm{lp}} = 0.29(6)$
SEC relaxation	$\sigma_c^{\mathrm{cp}} = 1.07(5)$	$b^{\mathrm{cp}} = 0.43(3)$
	$\sigma_c^{\tau} = 1.06(23)$	–
	$\sigma_c^{\mathrm{cp,g}} = 0.47(3)$	$b^{\mathrm{cp,g}} = 0.45(5)$
	$\sigma_c^{\tau,\mathrm{g}} = 0.44(8)$	–
	$\sigma_c^{\mathrm{cp,3}} = 1.18(8)$	$b^{\mathrm{cp,3}} = 0.40(4)$
fast Blossom rel.	$\sigma_c^{\mathrm{fb}} = 1.47(8)$	$b^{\mathrm{fb}} = 0.40(3)$